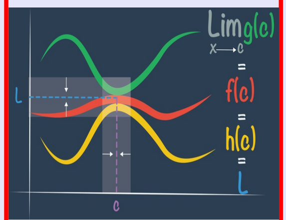
Math 261

Spring 2022

Lecture 7



Class QZ 3

Prove $\lim_{x\to 2}(3x+5)=11$.

$f(x)=3x+5$
$L=11$
$a=2$

$\lim_{x\to 2}(3x+5)=3(2)+5=11$ ✓

For $\boxed{\varepsilon>0}$, there is a $\boxed{\delta>0}$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

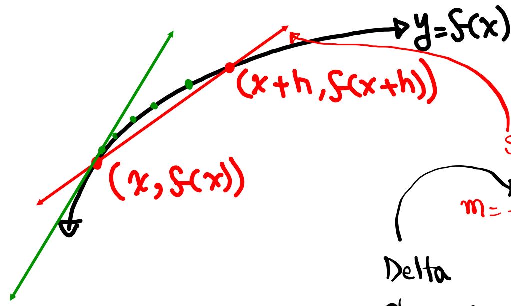
$|3x+5-11|<\varepsilon$ 〃 $|x-2|<\delta$

$|3x-6|<\varepsilon$ 〃 $|x-2|<\delta$

$|3(x-2)|<\varepsilon$ 〃 $|x-2|<\delta$

$3|x-2|<\varepsilon$ 〃 $|x-2|<\boxed{\delta}$

$|x-2|<\boxed{\frac{\varepsilon}{3}}$ So Pick $\boxed{\delta=\frac{\varepsilon}{3}}$ ✓



Secant line

$m = \frac{\Delta y}{\Delta x}$

Delta Changes

as $h \to 0$

Secant line $\to$ tangent line

$$\boxed{m_{\text{tan. line}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$\boxed{m_{\text{Secant line}} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}}$$

Slope of the tangent line to the graph of $y = f(x)$ at {any point} is called $\boxed{f\text{-Prime of } x}$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$\to$ First Derivative of $f(x)$

Given $f(x) = x^2 - 4x$, find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 4(x+h) - x^2 + 4x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-4x} - 4h \cancel{-x^2} \cancel{+4x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 4)}{\cancel{h}} = \lim_{h \to 0} (2x + h - 4)$$

$$= 2x + 0 - 4 = \boxed{2x - 4}$$

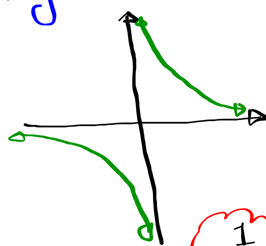
$$\boxed{\begin{aligned} F(x) &= x^2 - 4x \\ F'(x) &= 2x - 4 \end{aligned}}$$

March 2, 2022

Find $f'(x)$ using the limit for $f(x)=\frac{1}{x}$.

$f(x)=\frac{1}{x}$

Domain: $x\neq 0$



$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

LCD $=x(x+h)$

$$=\lim_{h\to 0}\frac{x-(x+h)}{h\,x(x+h)}=\lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}\,x(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}$$

$$=\frac{-1}{x(x+0)}=\boxed{\frac{-1}{x^2}}$$

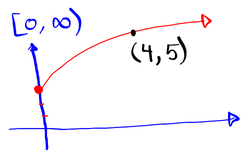
$$\boxed{\begin{aligned} f(x)&=\frac{1}{x}\\ f'(x)&=\frac{-1}{x^2}\end{aligned}}$$

Given $f(x)=\sqrt{x}+3$

Not Scaled

1) Domain: $x\geq 0$, $[0,\infty)$

2) Graph



3) $f(4)=\sqrt{4}+3=\boxed{5}$

4) find $f'(x)$ using the definition of limits.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}\cancel{+3}-\sqrt{x}\cancel{-3}}{h}$$

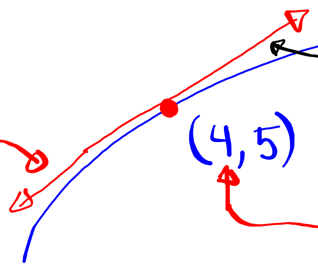
$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h\cancel{-x}}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}=\boxed{\frac{\sqrt{x}}{2x}}\quad x>0$$

$f(x)=\sqrt{x}+3$, $f'(x)=\frac{1}{2\sqrt{x}}$ $\quad f'(x)=\frac{\sqrt{x}}{2x}$

4) find $f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{2\cdot 2}=\boxed{\frac{1}{4}}$



$f(x) = \sqrt{x} + 3$

tan. line at $x=4$

$m_{\text{tan. line at } x=4} = f'(4) = \frac{1}{4}$

Now eqn of the tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x - 1 + 5$$

$$\boxed{y = \frac{1}{4}x + 4}$$

$f(x) = \sin x$, find $f'(x)$ using limits.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x [\cos h - 1]}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\to 0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\to 1}$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \boxed{\cos x}$$

$$\boxed{\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \end{aligned}}$$

March 2, 2022

Given $f(x) = x^3 - 4x$

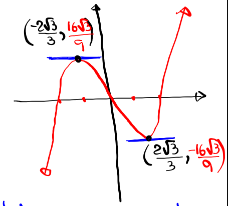
Domain: $(-\infty, \infty)$

$f(x) = x(x^2-4)$

$f(x) = x(x+2)(x-2)$

Y-Int $(0,0)$

X-Ints $(-2,0), (0,0), (2,0)$



Find all points where $f'(x) = 0 \Rightarrow$ Horizontal tangent line

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - 4(x+h) - x^3 + 4x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{4x} - 4h - \cancel{x^3} + \cancel{4x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 4)}{\cancel{h}} = \lim_{h\to 0}(3x^2 + 3xh + h^2 - 4)$$

$$= 3x^2 + 3x(0) + 0^2 - 4 = \boxed{3x^2 - 4}$$

$f(x) = x^3 - 4x$

$f'(x) = 3x^2 - 4$

$f'(x) = 0$

$3x^2 - 4 = 0$

$3x^2 = 4 \to x^2 = \frac{4}{3}$ $\quad x = \pm\sqrt{\frac{4}{3}}$ $\quad x = \pm\frac{2}{\sqrt{3}}$

$x = \pm \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \pm\frac{2\sqrt{3}}{\sqrt{9}}$ $\quad \boxed{x = \pm\frac{2\sqrt{3}}{3}}$

Find $f\left(\frac{2\sqrt{3}}{3}\right) = \left(\frac{2\sqrt{3}}{3}\right)^3 - 4\left(\frac{2\sqrt{3}}{3}\right)$

$$= \frac{8\cdot 3\sqrt{3}}{27} - \frac{8\sqrt{3}}{3} = \frac{24\sqrt{3}}{27} - \frac{8\sqrt{3}\cdot 9}{3\cdot 9}$$

$$= \frac{24\sqrt{3}}{27} - \frac{72\sqrt{3}}{27} = \frac{-\cancel{48}^{16}\sqrt{3}}{\cancel{27}_{9}} = \boxed{\frac{-16\sqrt{3}}{9}}$$

Prove that derivative of $\cos x$ is $-\sin x$.

Let $f(x) = \cos x$, use limit definition to find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$$= \lim_{h\to 0} \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\left[\frac{\cos x\cos h - \cos x}{h} - \frac{\sin x \sin h}{h}\right] =$$

$$= \cos x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} \to 0 \; - \sin x \cdot \lim_{h\to 0}\frac{\sin h}{h} \to 1$$

$$= \cos x \cdot 0 - \sin x \cdot 1 = \boxed{-\sin x}$$

$$\boxed{\begin{array}{l} f(x) = \cos x \\ f'(x) = -\sin x \end{array}}$$

March 2, 2022

Find an expression for the first derivative of $f(x)\cdot g(x)$

Let $H(x) = f(x)\cdot g(x)$

Find

$$H'(x) = \lim_{h\to 0}\frac{H(x+h) - H(x)}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\,g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(x+h)\left[f(x+h) - f(x)\right] + f(x)\left[g(x+h) - g(x)\right]}{h}$$

$$= \lim_{h\to 0} g(x+h)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$H(x) = \sin x \cos x$

$$H'(x) = \cos x \cdot \cos x + \sin x\cdot(-\sin x) = \cos^2 x - \sin^2 x = \boxed{\cos 2x}$$

Given $f(x) = x^n$, find $f'(x)$ using limit definition.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{x^n} + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 \cdots + h^n - \cancel{x^n}}{h}$$

h in common

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \cdots\cdots + h^n}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(nx^{n-1} + \underbrace{\cdots\cdots}_{\text{h in here}})}{\cancel{h}}$$

$$= \lim_{h\to 0}\left[nx^{n-1} + \cancel{\cdots}^{\,0}\right]$$

$$= n\,x^{n-1}$$

$F(x) = x^n$

$f'(x) = n x^{n-1}$

$$\boxed{\begin{aligned} f(x) &= x^5 \\ f'(x) &= 5x^4 \end{aligned}}$$

$$\boxed{\begin{aligned} f(x) &= x^{\frac{5}{2}} \\ f'(x) &= \tfrac{5}{2}x^{\frac{5}{2}-1} = \tfrac{5}{2}x^{\frac{3}{2}} \end{aligned}}$$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{1/2}}$

$$= \frac{1}{2\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{2x}$$

$$f(x) = \sqrt{x}$$
$$f'(x) = \frac{\sqrt{x}}{2x}$$

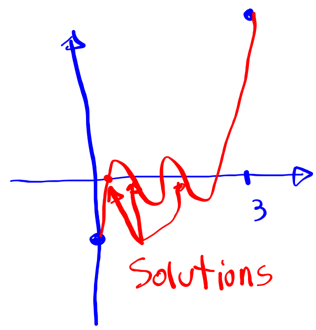
Show $2x^3 + 7x^2 - 3 = 0$ has a solution in the interval $[0, 3]$.

Polynomial expression $\Rightarrow$ Defined everywhere

Let $f(x) = 2x^3 + 7x^2 - 3$

$f(x)$ is cont. everywhere.

$f(0) = -3$, $f(3) = +$



By I.V.T., there is at least one solution in $(0, 3)$.

Evaluate

$$\lim_{x\to\infty} \frac{4x - 5}{\sqrt{x^2 - 1}}$$

Divide everything by highest power of x

Keep in mind as $x \to \infty$

$\sqrt{x^2 - 1} \approx \sqrt{x^2} = x$

$$= \lim_{x\to\infty} \frac{\frac{4x-5}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{4x}{x} - \frac{5}{x}}{\frac{\sqrt{x^2-1}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{4 - \frac{5}{x}}{\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{4 - \cancel{\frac{5}{x}}^{\to 0}}{\sqrt{1 - \cancel{\frac{1}{x^2}}^{\to 0}}} = \frac{4}{\sqrt{1}} = \boxed{4}$$

Evaluate $\lim_{x \to -\infty} \dfrac{\sqrt{4x^2+1}}{5-2x}$

$x \to -\infty$

$\sqrt{x^2} = -x \Rightarrow -\sqrt{x^2} = x$

$$= \lim_{x \to -\infty} \frac{\dfrac{\sqrt{4x^2+1}}{-\sqrt{x^2}}}{\dfrac{5-2x}{x}} = -\lim_{x \to -\infty} \frac{\sqrt{\dfrac{4x^2+1}{x^2}}}{\dfrac{5}{x} - \dfrac{2x}{x}}$$

$$= -\lim_{x \to -\infty} \frac{\sqrt{4 + \cancel{\frac{1}{x^2}}^{\to 0}}}{\cancel{\frac{5}{x}}^{\to 0} - 2} = -\frac{\sqrt{4}}{-2} = \frac{\sqrt{4}}{2} = \boxed{1}$$

Evaluate $\lim_{x \to \infty} \left(x - \sqrt{x^2 - 10x}\right) = \underbrace{\infty - \infty}_{\text{I.F.}}$

$$= \lim_{x \to \infty} \frac{(x - \sqrt{x^2-10x})(x + \sqrt{x^2-10x})}{x + \sqrt{x^2-10x}}$$

$$= \lim_{x \to \infty} \frac{x^2 - x^2 + 10x}{x + \sqrt{x^2-10x}} = \lim_{x \to \infty} \frac{10x}{x + \sqrt{x^2-10x}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{\dfrac{10x}{x}}{\dfrac{x}{x} + \sqrt{\dfrac{x^2-10x}{x^2}}} = \lim_{x \to \infty} \frac{10}{1 + \sqrt{1 - \cancel{\frac{10}{x}}^{\to 0}}}$$

$$= \frac{10}{2} = \boxed{5}$$

Class QZ 4

For $\varepsilon>0$, find $0<\delta\le 1$ such that

$\lim_{x\to -1}(x^2-3x)=4$

$f(x)=x^2-3x$

$L=4$ ✓

$a=-1$

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2-3x-4|<\varepsilon$ $\qquad$ $|x+1|<\delta$

$|(x-4)(x+1)|<\varepsilon$

$\underbrace{|x-4|}\,|x+1|<\varepsilon$

$|x+1|<1$

$-1<x+1<1$

$-2<x<0$

$-6<x-4<-4$

$-6<x-4<6$

$|x-4|<6$

$6\,|x+1|<\varepsilon$

$|x+1|<\dfrac{\varepsilon}{6}$

$\boxed{\text{Pick } \delta=\min\left\{1,\dfrac{\varepsilon}{6}\right\}}$